

TRIVIALIZING NUMBER OF POSITIVE KNOTS

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ABSTRACT. In this paper, we give the trivializing number of all minimal diagrams of positive 2-bridge knots and study the relation between the trivializing number and the unknotting number for a part of these knots.

1. INTRODUCTION

The trivializing number is one of numerical invariants of knots as same as the unknotting number, which judges some complexity of a knot. In general, it is known that the former is equal or over than twice the latter. Furthermore, Hanaki has conjectured that the trivializing number of a positive knot is just twice the unknotting number of the same knot. Indeed, for any positive knot up to 10 crossings, the equality such that $tr(K) = 2u(K)$ holds, where $tr(K)$ is the trivializing number of a knot and $u(K)$ is the unknotting number of the same knot (see [1]), and our result gives a partial positive answer of this conjecture.

This paper is constructed as follows: In Section 2, we define the trivializing number of a diagram and the trivializing number of a knot. In Section 3, we shortly do a review of positive knot and 2-bridge knot. In Section 4, we determine the standard diagrams of positive 2-bridge knots, and in Section 5, we determine the trivializing number of minimal diagrams of positive 2-bridge knots. In section 6, we show that for some positive 2-bridge knots, the equation $tr(K) = 2u(K)$ holds. In Section 7, we introduce positive pretzel knots, and show that the equation above also holds for some of them.

2. PRELIMINARIES

We work in the PL category. Throughout this paper, all knots are oriented. A *projection* of a knot K in \mathbb{R}^3 is a regular projection image of K in $\mathbb{R}^2 \cup \{\infty\} = S^2$. A *diagram* of K is a projection endowed with over/under information for its double points. A *crossing* is a double point with over/under information, and a *pre-crossing* is a double point without over/under information. A *pseudo-diagram* of K is a projection of K whose double points are either crossings or pre-crossings. See Figure 1.

A pseudo-diagram is said to be *trivial* if we always get a diagram of a trivial knot after giving arbitrary over/under information to all the pre-crossings. An example is given in Figure 2. It is known that we can change every projection into a trivial pseudo-diagram by giving appropriate over/under information to some of the pre-crossings.

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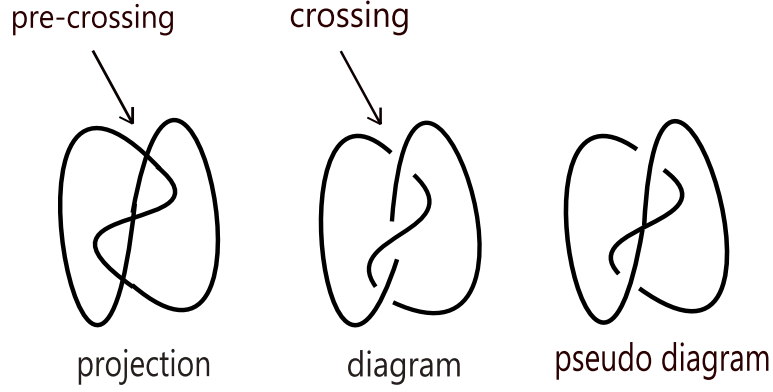


FIGURE 1. Projection, diagram, and pseudo-diagram

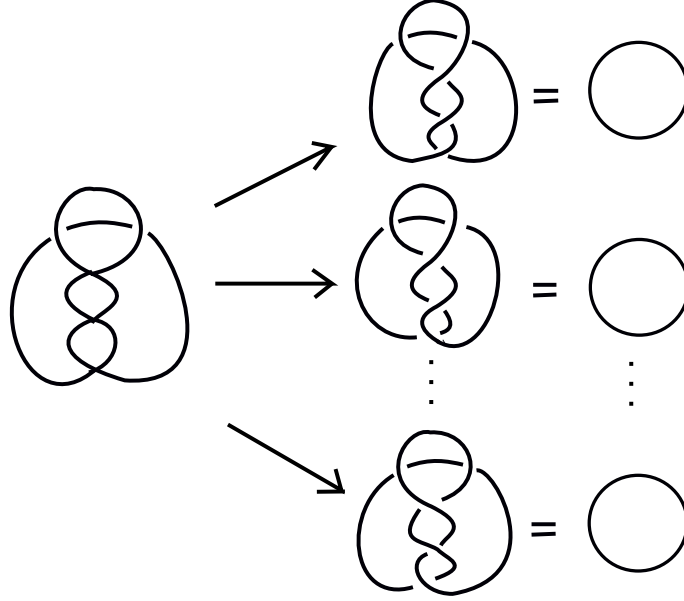


FIGURE 2. Example of a trivial pseudo-diagram

Definition 2.1. The *trivializing number* of a projection P , denoted by $\text{tr}(P)$, is the minimal number of pre-crossings of P which should be transformed into crossings for getting a trivial pseudo-diagram.

Definition 2.2. The *trivializing number* of a diagram D , denoted by $\text{tr}(D)$, is by definition the trivializing number of the associated projection which is obtained from D by ignoring the over/under information.

For example in the case as shown in Figure 3, we can get a trivial pseudo-diagram by transforming two pre-crossings ① and ② of P into crossings; however, it can be easily checked that we cannot get a trivial pseudo-diagram by transforming only one pre-crossing of P into a crossing. Therefore, we have $\text{tr}(D) = \text{tr}(P) = 2$.

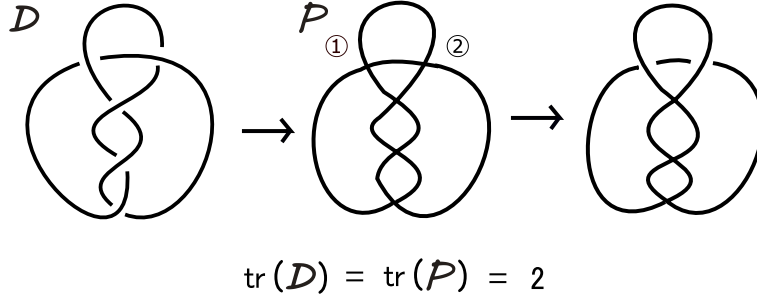


FIGURE 3. An operation for getting a trivial pseudo-diagram

Definition 2.3. The *trivializing number* of a knot K , denoted by $\text{tr}(K)$, is the minimum of $\text{tr}(D)$, where the minimum is taken over all diagrams D of K .

3. POSITIVE 2-BRIDGE KNOTS

Generally speaking, the trivializing number of a knot is not always realized by its minimal diagram (a diagram that has the minimal number of crossings); in fact we have counter examples (see [1]). Moreover, even for a given diagram, determining its trivializing number is not so easy in general. In Section 5, we give the trivializing numbers of all minimal diagrams of positive 2-bridge knots.

Let D be an oriented diagram of a knot. To each of its crossings, we associate sign $+$ or $-$ as shown in Figure 4(1). If all the crossings in D have the same sign $+$ (resp. $-$), then we say that D is a *positive diagram* (resp. *negative diagram*).

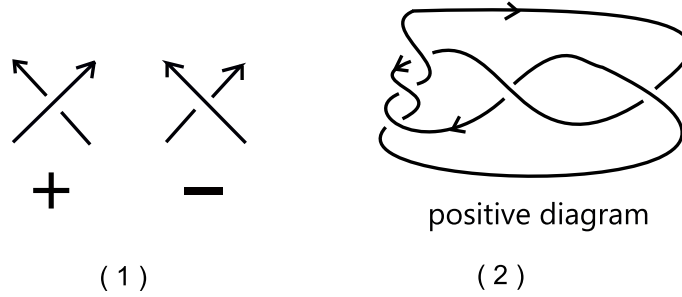


FIGURE 4. Sign of a crossing and an example of a positive diagram

When D is a positive diagram, the mirror image of D , which is obtained by changing the over/under information of all crossings of D and is denoted by D^* , is a negative diagram. Since D and D^* correspond to the same projection, we have $\text{tr}(D) = \text{tr}(D^*)$. A *positive knot* is a knot which has a positive diagram.

For a finite sequence a_1, a_2, \dots, a_m of integers, let us consider the knot (or link) diagram $D(a_1, a_2, \dots, a_m)$ as shown in Figure 5. In the figure, a rectangle in the upper row (resp. lower row), depicted by double lines (resp. simple lines), with integer a represents a left-hand (resp. right-hand) horizontal half-twists if $a \geq 0$, and $|a|$ right-hand (resp. left-hand) horizontal half-twists if $a < 0$. See Figure 6 for some explicit examples. We say a rectangle in the upper row (resp. lower row), an

upper rectangle (resp. a lower rectangle) for short. A knot which is represented by such a diagram is called a *2-bridge knot*.

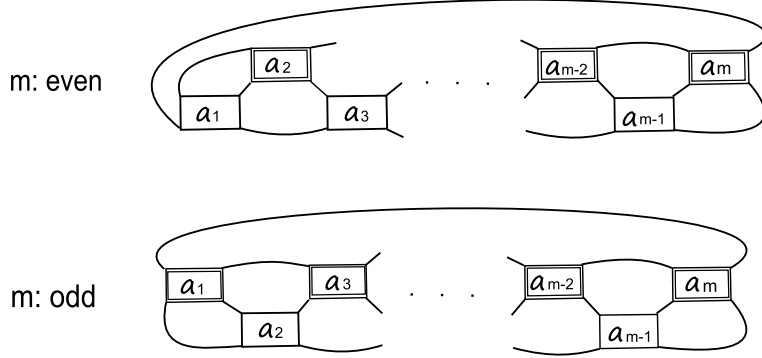


FIGURE 5. 2-bridge knot diagrams



FIGURE 6. Examples of 2-bridge knot diagrams

If $a_i > 0$ for all i with $1 \leq i \leq m$ or if $a_i < 0$ for all i , then the diagram $D(a_1, a_2, \dots, a_m)$ is reduced and alternating, and hence is a minimal diagram (see [2]). We call such a diagram a *standard diagram* of the knot.

It is known that every 2-bridge knot has a unique standard diagram (see, for example, [2]). Therefore, a positive (resp. negative) 2-bridge knot is a positive (resp. negative) alternating knot. A positive alternating knot may not necessarily have a diagram which is both positive and alternating in general. However, Nakamura has shown the following.

Theorem 3.1 (Nakamura [3]). *A reduced alternating diagram of a positive alternating knot is positive.*

By the theorem above, the standard diagram of a positive 2-bridge knot is necessarily positive.

In order to study the trivializing number of the standard diagram D of a positive or negative 2-bridge knot, by taking the mirror image, we may assume $a_i > 0$ for all i . Note that a positive knot may turn into a negative one by this operation.

4. STANDARD DIAGRAMS OF POSITIVE 2-BRIDGE KNOTS

In this section, we determine the standard diagrams of positive 2-bridge knots.

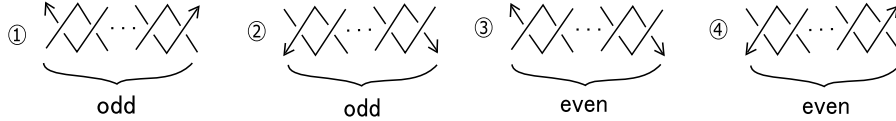
Proposition 4.1. *Let $D = D(a_1, a_2, \dots, a_m)$ be a positive diagram or a negative diagram of a 2-bridge knot such that $a_i > 0$ for all i with $1 \leq i \leq m$. Then D must be one of the following forms.*

- (1) When m is even, say $m = 2n$, we have either

- (a1) a_{2i} is even for $1 \leq i \leq n-1$,
- (a2) a_{2n} is odd, and
- (a3) $\sum_{i=1}^n a_{2i-1}$ is even,
- or
- (b1) a_1 is odd,
- (b2) a_{2i-1} is even for $2 \leq i \leq n$, and
- (b3) $\sum_{i=1}^n a_{2i}$ is even.
- (2) When m is odd, say $m = 2n + 1$, we have either
 - (a1) a_{2i-1} is even for $2 \leq i \leq n$,
 - (a2) a_1 and a_{2n+1} are odd, and
 - (a3) $\sum_{i=1}^n a_{2i}$ is odd,
 - or
 - (b1) a_{2i} is even for $1 \leq i \leq n$, and
 - (b2) $\sum_{i=1}^{n+1} a_{2i-1}$ is odd.

Let us consider a rectangle with integer $a_i > 0$, as in Figure 5, which corresponds to a_i left-hand (resp. right-hand) half-twists if it is in the upper (resp. lower) row. In the following, such a rectangle will sometimes be denoted by (a_i) . If its crossings all have the same sign $+$, then the orientation of the two arcs are of a form as in Figure 7. If the crossings all have sign $-$, then they are of a form as in Figure 8. Furthermore, we adopt the symbolic convention as depicted in Figure 9.

upper row



lower row

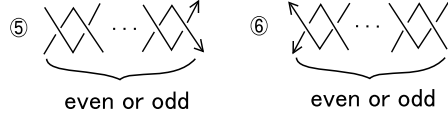


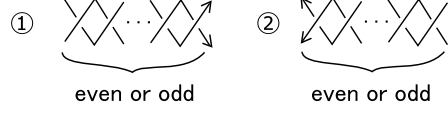
FIGURE 7. The orientations of two arcs with positive crossings

Proof of Proposition 4.1. (1) We may assume that the orientation of the diagram is as depicted in Figure 10, since it is a diagram of a knot.

When the crossings in (a_i) with $1 \leq i \leq 2n$ all have the same sign $+$, the orientations of the two arcs of (a_{2n}) are as shown in Figure 11 ① or ②. Then the orientation of the diagram is as shown in Figure 11 ④. Since the orientations of the arcs of (a_{2n-1}) must be as shown in Figure 11 ③, the orientations of the arcs of (a_{2n}) must be as in Figure 11 ①. In particular, a_{2n} is necessarily odd. Due to the orientation of (a_{2n-1}) , we see that the orientation of the other a_{2i-1} , $1 \leq i \leq n$ are of the form as in Figure 11 ③. Furthermore, by chasing the oriented arcs, we can determine the orientations of all arcs in the remaining rectangles. Hence, the oriented diagram is as depicted in Figure 12.

Then, we see that a_{2i} , $1 \leq i \leq n-1$, are all even. Since this is a knot diagram, the oriented strand passing through ① and then ② needs to pass through ③.

upper row



lower row

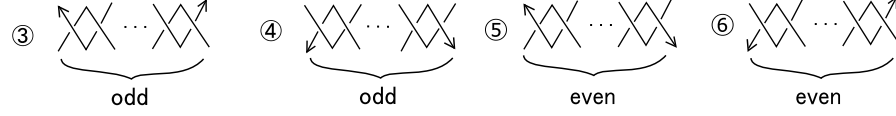


FIGURE 8. The orientations of two arcs with negative crossings

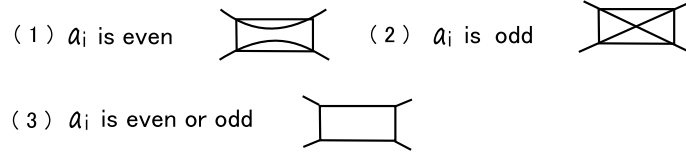


FIGURE 9. The symbolic convention

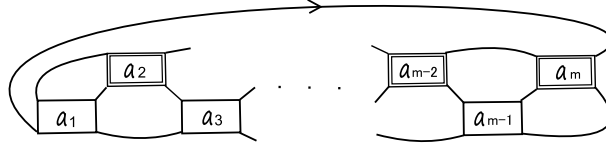


FIGURE 10. Orientation of diagram $D = D(a_1, a_2, \dots, a_m)$, m is even.

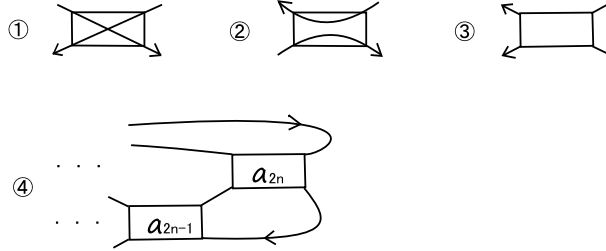


FIGURE 11. The orientation of arcs of rectangles

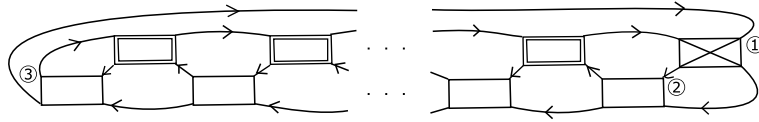


FIGURE 12. Orientations of arcs in D

Therefore, $\sum_{i=1}^n a_{2i-1}$ must necessarily be even. This shows that the conditions (a1), (a2) and (a3) are satisfied.

On the other hand, when the signs of crossings in (a_i) are all $-$, by turning the diagram as shown in Figure 13(1) on the plane by 180 degrees, and by reversing orientations of all arcs, we can get the mirror image of the diagram with the sign $+$ as shown in Figure 13(2). Consequently, we know that the conditions (b1), (b2) and (b3) are satisfied.

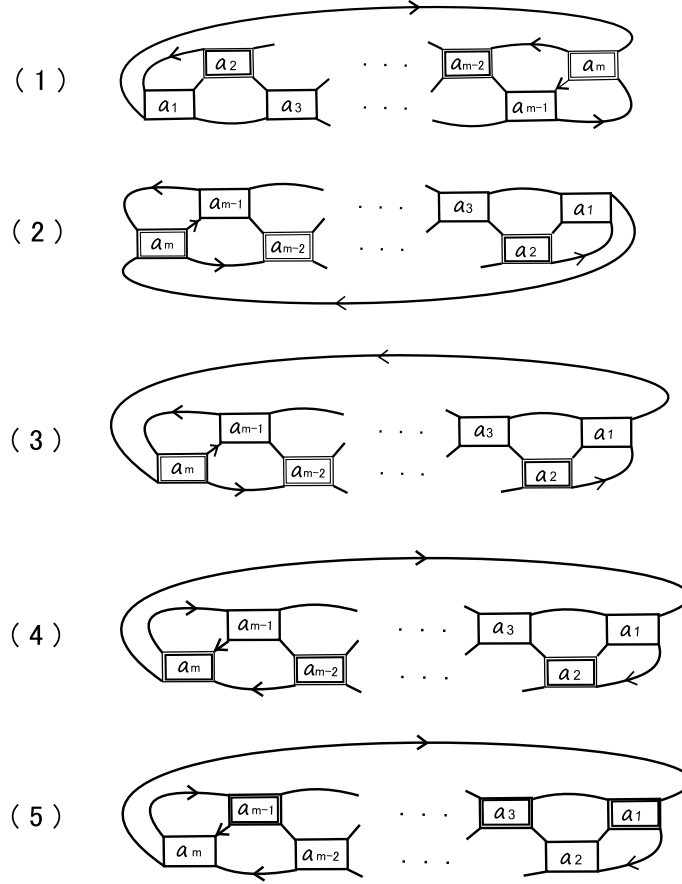


FIGURE 13. Orientations of arcs in D with all signs $-$

(2) When all crossings in (a_i) with $1 \leq i \leq 2n+1$ have the same sign $+$, we can consider in a similar way to the case where m is even. We may assume that the orientation of the diagram is as shown in Figure 14.

Since the orientations of the two arcs of (a_{2n+1}) are as shown in Figure 15 ① or ②, the orientation of the diagram is like as Figure 15 ④. Furthermore, the orientation of the arcs of (a_{2n}) is as shown Figure 15 ③, the orientation of the arcs of (a_{2n+1}) must be as shown in Figure 15 ①. In particular, a_{2n+1} is necessarily odd. Due to the orientation of (a_{2n}) , we see that the orientations other (a_{2i}) , $1 \leq i \leq n-1$ are of the form as in Figure 15 ③.

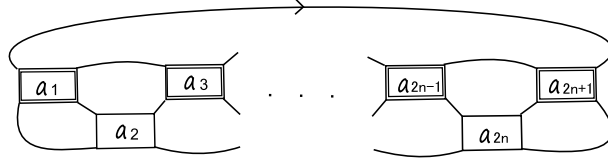
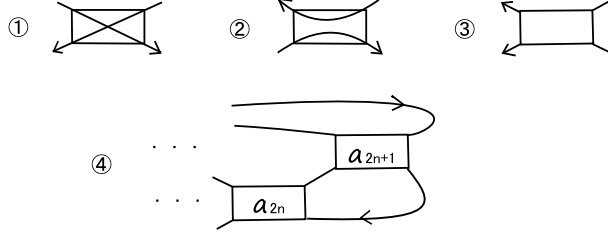
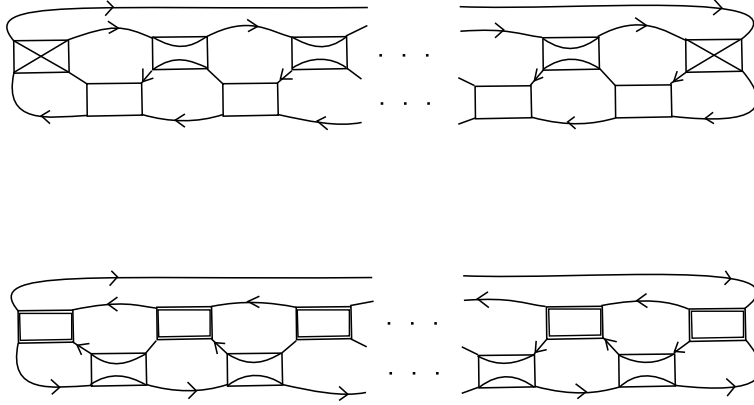
FIGURE 14. Orientation of diagram $D = D(a_1, a_2, \dots, a_m)$, m is odd,

FIGURE 15.

Hence, the oriented diagram is as depicted in Figure 16. This shows that the conditions (a1), (a2) and (a3) are satisfied.

FIGURE 16. The orientation of a positive diagram $D = D(a_1, a_2, \dots, a_m)$, m is odd

If the signs of the crossings in (a_i) are all negative, then the orientation of a_{2n+1} is as shown in Figure 17 ①. Therefore, the orientations of the other a_{2i+1} $1 \leq i \leq n-1$ must be of the same form as shown in Figure 17 ②. Thus the diagram is naturally as shown in Figure 17 ③, and we can easily see that the conditions (b1) and (b2) are satisfied. This completes the proof. \square

Beside, we classify these diagrams into four types, that is type of 1a, type of 1b, type of 2a, and type of 2b. Remark that these four types correspond not only to the proposition above but also to Main Theorem.

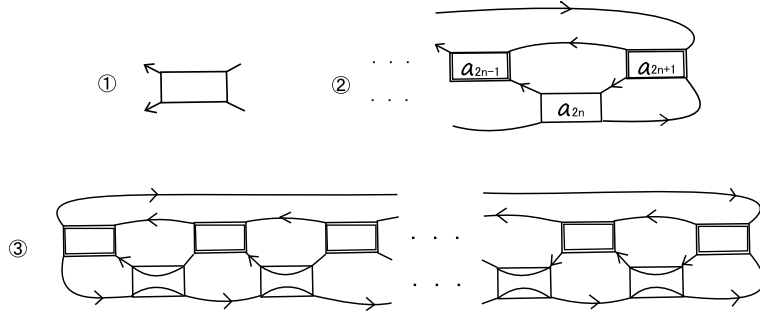


FIGURE 17. The orientation of a negative diagram $D = D(a_1, a_2, \dots, a_m)$, m is odd

5. MAIN THEOREM

For determining the trivializing number of a diagram, we can make use of the *chord diagram*. Let P be a projection with n pre-crossings. A *chord diagram* of P , denoted by CD_P , is a circle with n chords marked on it by dotted line segment where the preimage of each pre-crossing is connected by a chord (see [1]). We provide an example of the chord diagram as shown in Figure 18.

A chord diagram is said to be *parallel* if all the chords in it have no intersection. For example, the rightmost chord diagram in Figure 18 is made of the chords which correspond to the crossings 1, 2, 3, and is parallel.

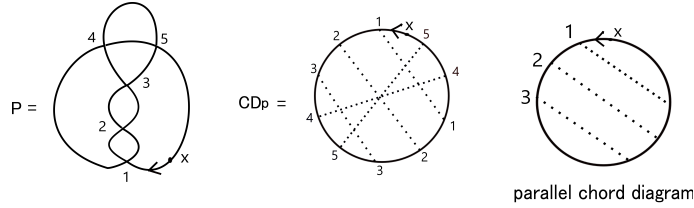


FIGURE 18. A chord diagram

In the situation above, next theorem holds.

Theorem 5.1 (Hanaki [1]). *For a chord diagram, the following holds.*

- The number of chords which are taken away from a chord diagram is even.
- $tr(D) = \min \{ \text{the number of chords which must be taken away from a chord diagram in order to get a parallel chord diagram} \}$

We consider sub chord diagram corresponding to each (a_i) .

Lemma 5.2. *Let SC_{a_i} be the sub-chord diagram corresponding to the rectangle (a_i) .*

- (1) *If two arcs of (a_i) enter from the same side, (left or right), as shown in Figure 19 ① and ②, then any two chords in SC_{a_i} certainly cross each other (i.e. any two chords have an intersection) as shown in Figure 19 ⑦.*

- (2) If one of two arcs enters from the left-hand side of (a_i) and the other enters from the right-hand side as shown in Figure 19 ③, ④, ⑤ and ⑥, then there are no intersections in SC_{a_i} as shown in Figure 19 ⑧. That is to say SC_{a_i} is parallel.

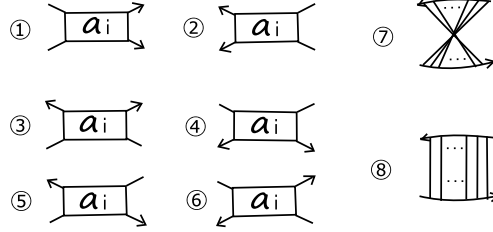


FIGURE 19. The orientations of two arcs in (a_i) and the sub-chord diagram corresponding to (a_i)

Proof. First we name the crossings in (a_i) , $1, 2, \dots, k$ from left to right.

- (1) If an arc enters from the left side, it passes the crossings $1, 2, \dots, k$ in order. Since the arc enters from the left side again, it passes the crossings in the same order. Therefore, the sub chord diagram corresponding to (a_i) is as shown in Figure 20(1).
- (2) On the otherhand, when the arcs enter from the different sides of (a_i) , the sub-chord diagram corresponding to (a_i) is naturally as shown in Figure 20(2).

□

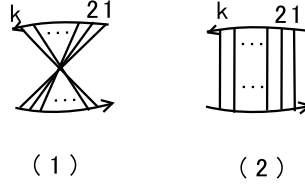


FIGURE 20. The sub-chord diagram corresponding to (a_i)

From the lemma above, we can consider a sub-chord diagram of a rectangle as one chord. That is to say, we can gather all chords in the sub-chord diagram corresponding to (a_i) into one chord denoted by \bar{a}_i . Furthermore, in the case of (1) in Lemma 5.2, we name this chord *I-chord* then represent it by a dotted line, while in the case of (2) we name this chord *P-chord* and represent it by a solid line.

Moreover, we determine the chord diagram CD_P corresponding to the diagram D . When D is of type 1a, by thinking over the orientation of each rectangle, we can see that any \bar{a}_{2i} for $1 \leq i \leq n$ is a P-chord, and any \bar{a}_{2i-1} for $1 \leq i \leq n$ is an I-chord. If every a_{2i-1} , $1 \leq i \leq n$, is even, then we obtain the diagram as shown in Figure 21(1), and also obtain the chord diagram as shown in Figure 21(2). (For convenience, we represent a chord diagram not by a circle but by a quadrangle.)

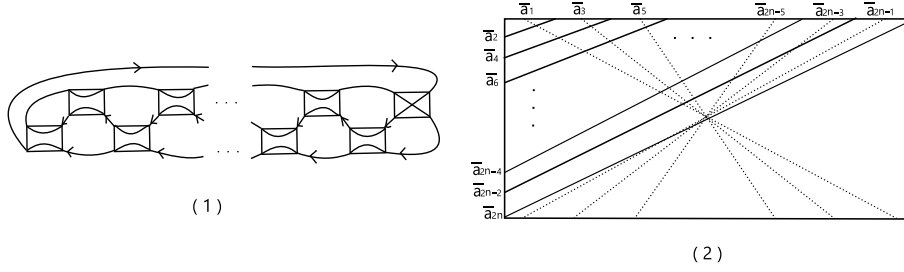


FIGURE 21. An oriented diagram D of type 1a with every a_{2i-1} even, and the chord diagram corresponding to D

Otherwise, the way to round a diagram depends on whether a_{2i-1} is odd or even. So we rename the lower rectangles which consist of odd number half-twists (the number of these tangles is even), $(b_1), (b_2), \dots, (b_{2r})$ from left to right in the diagram. Moreover we also rename all upper rectangles as following:

- the upper rectangles on the left-hand side of (b_1) ; $(c_0^1), (c_0^2), \dots, (c_0^{q_0})$
- the upper rectangles between (b_j) and (b_{j+1}) ; $(c_j^1), (c_j^2), \dots, (c_j^{q_j})$
- the upper rectangles on the right-hand side of (b_{2r}) ; $(c_{2r}^1), (c_{2r}^2), \dots, (c_{2r}^{q_{2r}})$

Then we can obtain the sub chord diagram corresponding to the rectangles between (b_j) and (b_{j+1}) as shown in Figure 22, where $\overline{c_j^k}$ ($1 \leq k \leq q_j$) consists of some parallel chords which correspond to the crossings in rectangle (c_j^k) .

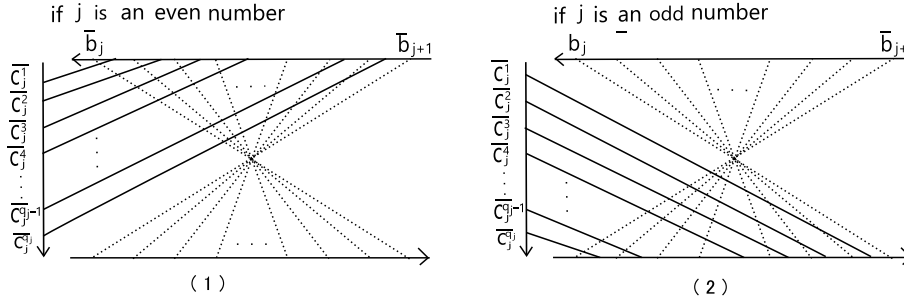


FIGURE 22. The sub chord diagram between (b_i) and (b_{j+1})

Furthermore, any two P-chords in $\overline{c_j^1}, \overline{c_j^2}, \dots, \overline{c_j^{q_j}}$ does not cross each other, so we can bundle them again into one P-chord. Now we represent them by a solid line. In other words, we consider that $c_j = c_j^1 + c_j^2 + \dots + c_j^{q_j}$. Since the chords which correspond to the lower tangles between (b_j) and (b_{j+1}) are all I-chords, by Theorem 5.1 the number of chords which we can leave is at most only one. Besides, the I-chord which does not cross $\overline{c_j}$ is only $\overline{b_j}$ as shown in Figure 22(1) or $\overline{b_{j+1}}$ as shown in Figure 22(2). So we only need to consider the chord diagram in which all I-chords between $\overline{b_j}$ and $\overline{b_{j+1}}$ were already deleted as shown in Figure 23.

In the case of the diagram of type 1b, we can consider in a similar fashion to type 1a. When every a_{2i} is even, the diagram is as shown in Figure 24(1), and

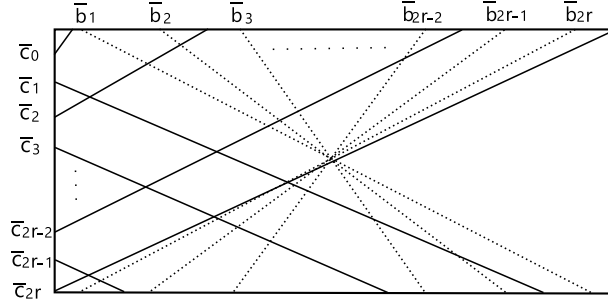


FIGURE 23. The chord diagram corresponding to the diagrams of type 1a.

the chord diagram is as shown in Figure 24(2). Otherwise, the sub chord diagram between (b_j) and (b_{j+1}) is also as shown in Figure 22, and we can get the chord diagram as shown in Figure 25.

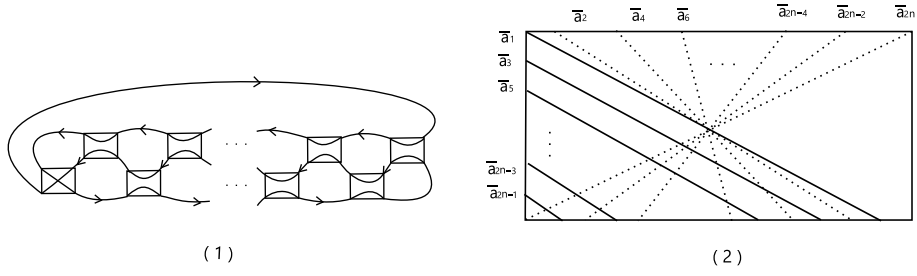


FIGURE 24. An oriented diagram D of type 1b with every a_{2i} even, and the chord diagram corresponding to D

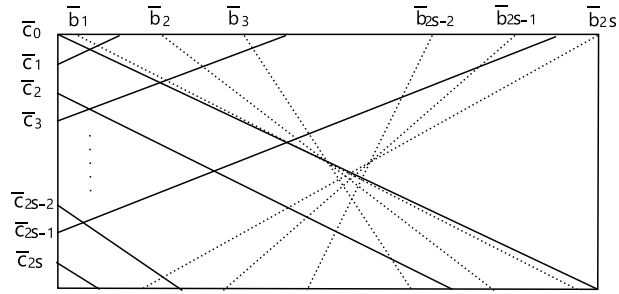


FIGURE 25. The chord diagram corresponding to the diagrams of type 1b.

On the other hand, in the case of the diagrams of type 2a and type 2b, when j is an even number, the sub chord diagram is as shown in Figure 22(1), and when j is an odd number, the sub chord diagram is as shown in Figure 22(2). Thus we can get the chord diagrams as shown in Figure 26 and Figure 27.

In the condition above, we can obtain the theorem bellow.

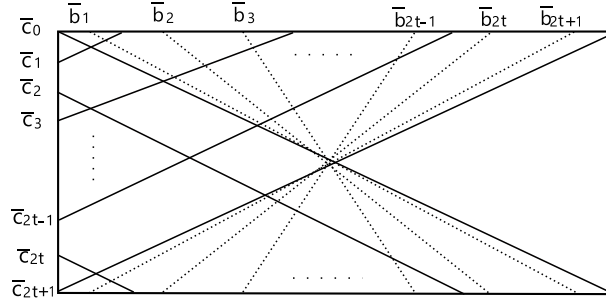


FIGURE 26. The chord diagram corresponding to the diagrams of type 2a.

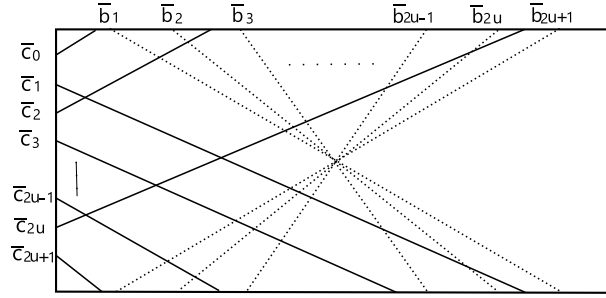


FIGURE 27. The chord diagram corresponding to the diagrams of type 2b.

Theorem 5.3. Let $D = D(a_1, a_2, \dots, a_m)$ be a positive diagram or a negative diagram of a 2-bridge knot such that $a_i > 0$ for all i with $1 \leq i \leq m$. Then for the trivializing number of D , the following holds.

- (1) When D is of type 1a.
 - (a) If every a_{2i-1} is even, then $tr(D) = \sum_{i=1}^n a_{2i-1}$.
 - (b) Otherwise,

$$tr(D) = \min \left\{ \sum_{i=1}^n a_{2i-1} + \sum_{j=1}^r c_{2j-1}, \sum_{i=1}^n a_{2i-1} + \sum_{j=1}^p c_{2j-1} + \sum_{j=p+1}^r c_{2j} - 1 \right\}$$
- (2) When D is of type 1b.
 - (a) If every a_{2i} is even, then $tr(D) = \sum_{i=1}^n a_{2i}$.
 - (b) Otherwise,

$$tr(D) = \min \left\{ \sum_{i=1}^n a_{2i} + \sum_{j=0}^p c_{2j} + \sum_{j=p+2}^s c_{2j-1} - 1, \sum_{i=1}^n a_{2i} + \sum_{j=1}^s c_{2j-1} \right\}$$
- (3) When D is of type 2a.

$$tr(D) = \min \left\{ \sum_{i=1}^n a_{2i} + \sum_{j=0}^t c_{2j+1}, \sum_{i=1}^n a_{2i} + \sum_{j=0}^p c_{2j} + \sum_{j=p+1}^t c_{2j+1} \right\}$$
- (4) When D is of type 2b.

$$tr(D) = \min \left\{ \sum_{i=1}^n a_{2i+1} + \sum_{j=0}^p c_{2j+1} + \sum_{j=p+2}^u c_{2j} - 1, \sum_{i=1}^n a_{2i+1} + \sum_{j=0}^u c_{2j+1} \right\}$$

Proof 5.4. (1) When D is of type 1a.

- (a) If every a_{2i-1} is even, then we have the chord diagram as shown in Figure 21(2). Since any two I-chords in this chord diagram cross each

other, we can leave at most only one I-chord when we attempt to gain a trivial chord diagram. Moreover every two chords in any I-chord also cross each other. This means that the number of the chords corresponding to the crossings in lower rectangles which we can leave is at most only one.

In addition, the P-chords corresponding to the crossings in upper rectangles are all parallel and any P-chord crosses at least one I-chord. Hence the minimal number of the chords which we must delete in order to get a trivial chord diagram is the number of all the chords which correspond to the crossings in lower rectangles. Therefore, we have the following:

$$tr(D) = \sum_{i=1}^n a_{2i-1}$$

- (b) Otherwise, the chord diagram as shown in Figure 23, and we can see the P-chord represented by \bar{c}_{2r} crosses all P-chords represented by \bar{c}_{2j-1} ($1 \leq j \leq r$) and all I-chords represented by \bar{b}_k ($1 \leq k \leq r$). (This means that \bar{c}_{2r} crosses all chords corresponding to the crossings in lower rectangles). So if we leave \bar{c}_{2r} then we must delete all these chords which cross \bar{c}_{2r} . That is to say the number of chords we must delete is $\sum_{i=1}^n a_{2i-1} + \sum_{j=1}^r c_{2j-1}$. See Figure 28(1).

When we delete \bar{c}_{2r} , we can leave all chords in P-chord \bar{c}_{2r-1} and only one chord in I-chord \bar{b}_{2r-1} . So the number of chords we need to delete is $\sum_{i=1}^n a_{2i-1} + \sum_{j=1}^{r-1} c_{2j-1} + c_{2r} - 1$. See Figure 28(2).

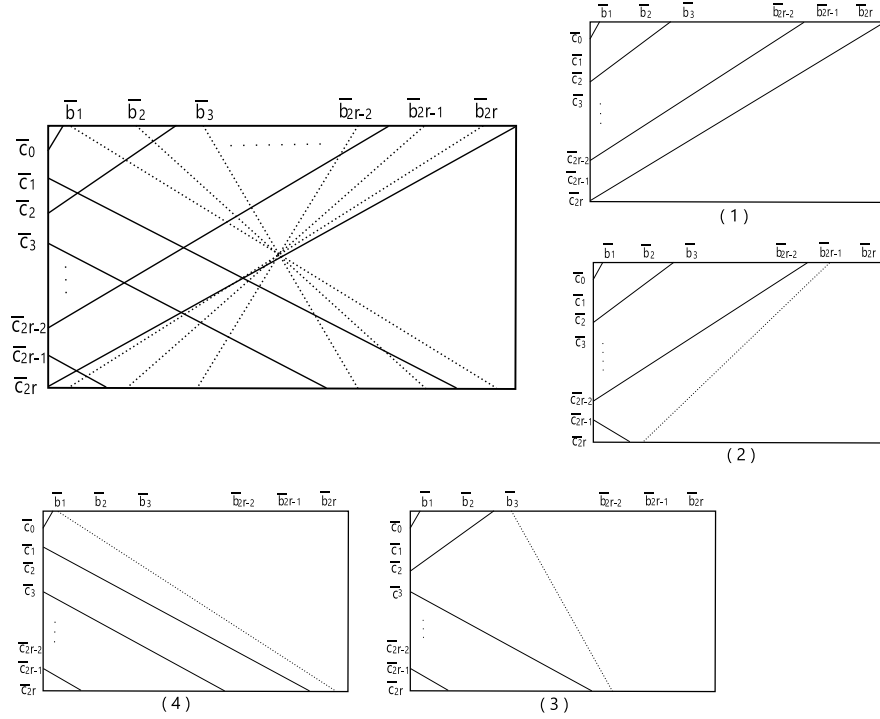


FIGURE 28. The operation of deleting some P-chords

Next we attempt to delete the P-chords which correspond to \bar{c}_{2j} ($1 \leq j \leq r$) step by step in the way as following: $\{\bar{c}_{2r}\} \rightarrow \{\bar{c}_{2r}, \bar{c}_{2r-2}\} \rightarrow \{\bar{c}_{2r}, \bar{c}_{2r-2}, \bar{c}_{2r-4}\} \rightarrow \cdots \rightarrow \{\bar{c}_{2r}, \bar{c}_{2r-2}, \dots, \bar{c}_2\}$.

By these operations we can also get a trivial chord diagram even if we leave the P-chords which correspond to \bar{c}_{2j-1} ($1 \leq j \leq r$) step by step in the way as following: $\{\bar{c}_{2r-1}\} \rightarrow \{\bar{c}_{2r-1}, \bar{c}_{2r-3}\} \rightarrow \{\bar{c}_{2r-1}, \bar{c}_{2r-3}, \bar{c}_{2r-5}\} \rightarrow \cdots \rightarrow \{\bar{c}_{2r-1}, \bar{c}_{2r-3}, \dots, \bar{c}_1\}$.

There is an one-to-one correlation between these two operations. Consequently, the minimum of these numbers is the trivializing number of the diagram, and the following holds.

$$tr(D) = \min \left\{ \begin{array}{l} \sum_{i=1}^n a_{2i-1} + \sum_{j=1}^r c_{2j-1} \\ \sum_{i=1}^n a_{2i-1} + \sum_{j=1}^p c_{2j-1} + \sum_{j=p+1}^r c_{2j} - 1 \end{array} \right\}$$

(2) When D is of type 1b.

- (a) If every a_{2i} is even, then we can consider in a similar fashion to type 1a and can easily see $tr(D) = \sum_{i=1}^n a_{2i}$.
- (b) Otherwise, from the chord diagram as shown in Figure 25, we know the P-chord \bar{c}_{2r} in Figure 23 is replaced by \bar{c}_0 in Figure 25. In this case, if we delete the P-chords represented by \bar{c}_{2j} ($0 \leq j \leq s$) step by step in the way $\{\bar{c}_0\} \rightarrow \{\bar{c}_0, \bar{c}_2\} \rightarrow \{\bar{c}_0, \bar{c}_2, \bar{c}_4\} \dots$, then we can leave $\{\bar{c}_1\} \rightarrow \{\bar{c}_1, \bar{c}_3\} \rightarrow \{\bar{c}_1, \bar{c}_3, \bar{c}_5\} \dots$, by way of compensation. Thus the following holds.

$$tr(D) = \min \left\{ \begin{array}{l} \sum_{i=1}^n a_{2i} + \sum_{j=1}^s c_{2j-1} \\ \sum_{i=1}^n a_{2i} + \sum_{j=0}^p c_{2j} + \sum_{j=p+2}^s c_{2j-1} - 1 \end{array} \right\}$$

(3) When D is of type 2a. In this case, the chord diagram is as shown in Figure 26, and we see that every I-chord which corresponds to (b_j) ($1 \leq j \leq 2t+1$) necessarily crosses two P-chords which correspond to (c_0) and (c_{2t+1}) . So we can leave none of these I-chords unless we delete both \bar{c}_0 and \bar{c}_{2t+1} . In addition, we consider the relation of P-chords which correspond to (c_j) ($1 \leq j \leq 2t+1$). If we leave every \bar{c}_{2j} ($0 \leq j \leq t$), then we must delete every \bar{c}_{2j+1} ($0 \leq j \leq t$). Hence the number of chords which we need to delete is the following:

$$\sum_{i=1}^n a_{2i} + \sum_{j=0}^t c_{2j+1}.$$

Furthermore there exists a relation in the P-chords in this chord diagram.

That is, if we delete \bar{c}_0 then we can leave \bar{c}_1 , if we delete $\{\bar{c}_0, \bar{c}_2\}$ then we can leave $\{\bar{c}_1, \bar{c}_3\}$, and so on. Because of this, the following holds.

$$tr(D) = \min \left\{ \begin{array}{l} \sum_{i=1}^n a_{2i} + \sum_{j=0}^t c_{2j+1} \\ \sum_{i=1}^n a_{2i} + \sum_{j=0}^p c_{2j} + \sum_{j=p+1}^t c_{2j+1} \end{array} \right\}$$

(4) When D is of type 2b, the chord diagram is as shown in Figure 27. In this chord diagram, \bar{c}_0 and \bar{c}_{2u+1} dose not cross each other. Moreover they dose not cross any other P-chord or I-chord. Therefore, we can leave both \bar{c}_0 and \bar{c}_{2u+1} . However for I-chords \bar{b}_j ($1 \leq j \leq 2u+1$), any two of them cross each other, so we can leave at most only one I-chord among $\{\bar{b}_j\}$. Thus, if we leave all P-chords corresponding to (c_{2k}) ($1 \leq k \leq u$), we must delete all P-chords corresponding to (c_{2k-1}) ($1 \leq k \leq u$). Hence the number of all chords which we must delete is $\sum_{i=1}^n a_{2i+1} + \sum_{j=0}^{u-1} c_{2j+1} - 1$.

Besides, if we orderly delete some P-chords step by step such as $\{\bar{c}_{2u}\} \rightarrow \{\bar{c}_{2u}, \bar{c}_{2u-2}\} \rightarrow \{\bar{c}_{2u}, \bar{c}_{2u-2}, \bar{c}_{2u-4}\} \dots$, we can leave other P-chords such as

$\{\bar{c}_{2u-1}\} \rightarrow \{\bar{c}_{2u-1}, \bar{c}_{2u-3}\} \rightarrow \{\bar{c}_{2u-1}, \bar{c}_{2u-3}, \bar{c}_{2u-5}\} \cdots$ by way of compensation. Finally the following holds.

$$tr(D) = \min\{\sum_{i=1}^n a_{2i+1} + \sum_{j=0}^p c_{2j+1} + \sum_{j=p+2}^u c_{2j} - 1\}$$

We have just completed the proof of Main Theorem. \square

6. THE RELATION BETWEEN TRIVIALIZING NUMBER AND UNKNOTTING NUMBER

In this section we study the relation between the trivializing number and the unknotting number. The definitions of the unknotting number of a diagram and the unknotting number of a knot are the following:

Definition 6.1. The *unknotting number* of a diagram D , denoted by $u(D)$, is the minimal number of crossings of D whose over/under information should be changed for getting a diagram of a trivial knot.

Definition 6.2. The *unknotting number* of a knot K , denoted by $u(K)$, is the minimum of $u(D)$, where the minimum is taken over all diagrams D of K .

There is a relation between the unknotting number and the signature. (About the signature there is a detailed explanation in [2]). The signature is an invariant of knots, and in general the following holds.

Theorem 6.3 ([2]). $\frac{1}{2}|\sigma(D)| = \frac{1}{2}|\sigma(K)| \leq u(K) \leq u(D)$

In addition for a alternating diagram, it is known that $\sigma(D) = -w(D)/2 + (W - B)/2$ ([4]), where $w(D)$ is the sum of local writhes of all crossings, B is the number of domains colored with a grayish color when we give checkerboard coloring as shown in Figure 29, and W is the number of domains which are not colored. For example, in the case as shown in Figure 29, the number of $+$ crossings is 2, and that of $-$ crossings is 4, then $\sigma(D) = 2 + (-4) = -2$, and $W = 5$, $B = 3$. Therefore we can get $\sigma(D) = \sigma(K) = -(-2)/2 + (5 - 3)/2 = 2$, and $|\sigma(D)|/2 = 1 \leq u(K) \leq u(D)$. In actually, we can obtain a diagram of a trivial knot with one crossing change, hence $u(D) = u(K) = 1$.



FIGURE 29. An example of the checkerboard coloring and local writhes

About the relation between the trivializing number and the unknotting number, it is known that $2u(D) \leq tr(D)$ ($2u(K) \leq tr(K)$) holds in general. However, particularly for positive knots, there exists a conjecture that $2u(K) = tr(K)$ ([1]). And as the partial positive answer of this, we have the next corollary to Theorem 5.3 and Theorem 6.3.

Corollary 6.4. Let K be a positive 2-bridge knot and has a diagram $D = C(a_1, a_2, \dots, a_{2n})$ $a_i > 0$ for any i ($1 \leq i \leq 2n$). If a_{2i-1} is even for any i ($1 \leq i \leq n$), or a_{2i} is even for any i ($1 \leq i \leq n$), then $2u(K) = tr(K)$.

Proof. First we prove the case where any a_{2i-1} is an even number. In this case, D is a minimal diagram of K , so by Theorem 3.1, D is an positive and alternating diagram. Besides, by the Proposition 4.1, a_{2n} must be an odd number and other a_{2i} ($1 \leq i \leq n-1$) are necessarily all even numbers. Moreover, the sign of any crossing is $+$, thereby $w(D) = \sum_{i=1}^{2n} a_i$. The checkerboard coloring is like as shown in Figure 30, and we know $W = \sum_{i=1}^n a_{2i} + 1$, $B = \sum_{i=1}^n a_{2i-1} + 1$.



FIGURE 30. The checkerboard coloring of a 2-bridge knot diagram in which any a_{2i-1} is an even number

Therefore, next equality holds.

$$\sigma(D) = -\frac{1}{2}(w(D)) + \frac{1}{2}(W - B) = \frac{1}{2}(-\sum_{i=1}^{2n} a_i + \sum_{i=1}^n a_{2i} - \sum_{i=1}^n a_{2i-1}) = -\sum_{i=1}^n a_{2i-1}$$

Furthermore by Theorem 6.3, we can see $(|\sigma(D)|)/2 = (\sum_{i=1}^n a_{2i-1})/2 \leq u(K) = u(D)$. In actually as shown in Figure 31, we can obtain a trivial diagram by some crossing changes of the crossings which correspondent to lower tangles, and the number of these crossing changes is $(\sum_{i=1}^n a_{2i-1})/2$. Hence $u(D) = u(K) = (\sum_{i=1}^n a_{2i-1})/2$.

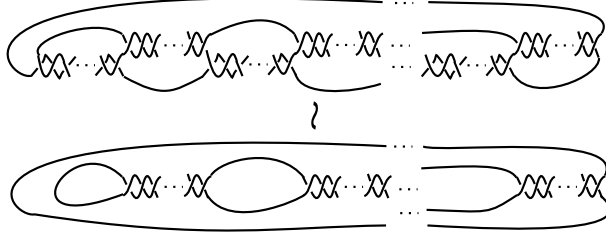


FIGURE 31. By some crossing changes, we can obtain a trivial knot diagram.

Finally we can get the inequality $2u(K) \leq tr(K) \leq tr(D)$ and the equality $2u(K) = tr(D)$. Thus $2u(K) = tr(K)$ holds.

In the case that any a_{2i} is an even number, we can also gain this equality in a similar fashion. □

This result is for the special case of positive 2-bridge knots. So whether $2u(D) = tr(D)$ holds for any minimal diagram of positive 2-bridge knots or not, and whether $2u(K) = tr(K)$ holds or not, these questions are our theme of the future.

7. POSITIVE PRETZEL KNOT

For positive pretzel knots we can get the following:

Theorem 7.1. *Let K be a pretzel knot $P(p_1, p_2, \dots, p_{2n})$ $p_i > 0$ for any i ($1 \leq i \leq 2n$), p_{2n} is even and other p_i s are all odd ($1 \leq i \leq 2n-1$), then the following holds.*

$$tr(K) = 2u(K) = \sum_{i=1}^{2n} p_i - 2n + 1$$

Proof. A diagram D of knot K is as shown in Figure 32, and we know this diagram is positive and alternating. Besides the sub-chord diagram which corresponds to each \bar{p}_i is an I-chord. So the chord diagram of D is as shown in Figure 33.

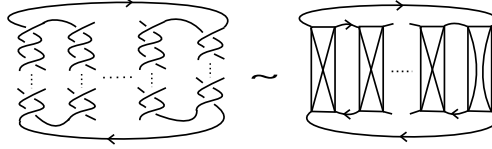


FIGURE 32. The standard diagram D of K

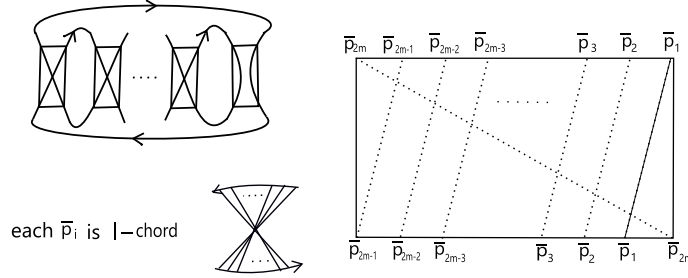


FIGURE 33. The chord diagram of K

Then we can easily obtain the trivializing number of D . Namely, $tr(D) = \sum_{i=1}^{2n} p_i - 2n + 1$.

Furthermore, by the checkerboard coloring as shown in Figure 34, the signature of K is the following:

$$\sigma(K) = \sigma(D) = -\frac{1}{2}w(D) + \frac{1}{2}(W - B) = -(\sum_{i=1}^{2n} p_i - 2n + 1)$$

By the inequality $|\sigma(K)| \leq 2u(K) \leq tr(K) \leq tr(D)$, we can conclude $tr(K) = 2u(K)$.

This completes the proof.

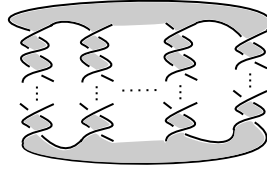


FIGURE 34. An example of checkerboard coloring

□

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